JOURNAL OF APPROXIMATION THEORY 46, 204–212 (1986)

Norm Oscillatory Weight Measures

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Communicated by. V. Totik

Received December 3, 1984; revised January 18, 1985

DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION

When one considers orthogonal polynomials for general measures, several phenomena occur that have no analogue in the classical prototype of Jacobi polynomials. This paper is concerned with showing the existence of oscillatory linearized norm behavior, and in showing that, when the linearized norm oscillates, the range of its limit points is an interval. The final section contains a discussion of the implication of these results for zero distribution, and also contains, in Theorem 5.2, a related result due to the second author, whose proof will appear elsewhere.

It is especially appropriate to dedicate this paper to the memory of Professor Géza Freud who interacted in many positive ways with the first and third authors and in particular was co-discoverer with L. Ziegler of Theorem 2.2 of this paper.

2. DEFINITIONS AND STATEMNET OF THEOREMS

DEFINITION 2.1. Let μ be a unit measure defined on the Borel subsets of $I=[-1,1]$, whose support $S(\mu)$ is an infinite set. The unique polynomials $\{P_n(x)\}$ or $\{P_n(x,\mu)\}$, $P_n(x)=x^n+\cdots$, and the unique constants $\{N_n(\mu)\}$, $n=0,1,\ldots$, such that $\int P_m(x)\,P_n(x)\,d\mu=\delta_{m,n}(N_n(\mu))^2$, $n,m=0,1,\ldots$, where $\delta_{n,m}=0$ if $n\neq m$ and 1 if $n=m$, are called the orthogonal

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0021-9045/86 $3.00

polynomials and their norms for the weight measure μ. We also let $\lambda_n(\mu) = (N_n(\mu))^{1/n}$ and call it the linearized norm.

DEFINITION 2.2. For a compact set $K \subset I$, we denote the logarithmic capacity by $C(K)$ [5, p. 55], and henceforth refer to this as the capacity of K. For a general set $E \subset I$, $C(E)$ is defined as the inner capacity, and is also referred to as the capacity of E.

DEFINITION 2.3. Let μ be a weight measure with support $S(\mu)$. A Borel set $E \subset S(\mu)$ for which $\mu(E) = 1$ is called a carrier of μ. Let $\bar{C} = C(S(\mu))$ and let $\underline{C} = \operatorname{Inf} C(E)$, where E ranges over the carriers of μ. These numbers are referred to as the upper and lower carrier capacities of μ. If $\underline{C} < \bar{C}$ we call μ an undetermined weight measure. Otherwise it is called a determined weight meausure. See [6, p. 121] for proof of the existence of undetermined weight measures.

DEFINITION 2.4. Let μ be a weight measure. Another weight measure v is said to be carrier related to μ, written $v \sim \mu$, if every carrier of μ is a carrier of v, and every carrier of v is a carrier of μ. It is clear that the relation $\sim$ is an equivalence relation in the class of weight measures.

THEOREM 2.1. *Let μ be an undetermined weight measure with lower and upper carrier capacities $\underline{C}$ $\bar{C}$. Let C_1, C_2 be two numbers that satisfy $\underline{C} < C_1 < C_2 < \bar{C}$. Then there is a weight measure v, carrier related to μ, such that* $\underline{\lim}_{n \to \infty} \lambda_n(v) \leqslant C_1$ and $\overline{\lim}_{n \to \infty} \lambda_n(v) \geqslant C_2$.

DEFINITION 2.5. A weight measure μ for which the linearized norm $\lambda_n(\mu)$ does not converge is called a norm oscillatory weight measure. Otherwise μ is called a norm convergent weight measure.

THEOREM 2.2. *Let μ be a norm oscillatory weight measure. Then the limit points of the sequence of linearized norms form an interval.*

Since by [6, p. 121] undetermined weight measures exist, by Theorem 2.1 norm oscillatory weight measures exist. However, we will show that an undetermined weight measure need not be norm oscillatory, and we will study further properties of undetermined, norm convergent weight measures in a forthcoming paper.

3. PROOF OF THEOREM 2.1

We first assemble needed lemmas and definitions, then sketch the proof and finally complete the details.

LEMMA 3.1 [7]. *Let μ be a weight measure. Then a measure v is a* *weight measure and satisfies* $\nu \sim \mu$ *if there is a Borel measurable function* $w(x)$, *positive* a.e. μ, *such that* $\int w(x)\, d\mu = 1$ *and is such that for any Borel set* $E \subset I$, $\nu(E) = \int_E w(x)\, d\mu$. *When* ν *is related to* μ *in this way, we use the notation* $\nu = \mu_w$.

LEMMA 3.2. *Let* μ *be a weight measure in the undetermined case, with* $\underline{C}$ $\bar{C}$ *as the lower and upper carrier capacities. If* C_1 *is a real number satisfying* $\underline{C} < C_1 < \bar{C}$, μ *has a carrier* E *with the representations* $E = \bigcup_{n=1}^{\infty} E_n$, *where* E_n *is compact,* $E_n \subset E_{n+1}$, $\mu(E_n) > 0$, $n = 1, 2, \ldots$, E *is not compact and* $C(E) < C_1$.

PROOF OF LEMMA 3.2. *There is a carrier, say* E^*, *of capacity less than* C_1 *by the definition of* $\underline{C}$. *Since* μ *is a normal measure,* E^* *contains compact sets whose* μ *measure is as close to one as desired. Hence a carrier* E *with the stated structure can be achieved as a subset of* E^*. *Since* $C(E) \leqslant C(E^*) < C_1 < \bar{C}$, E *cannot be compact, since* $S(\mu)$ *is the smallest compact carrier of* μ.

DEFINITION 3.1. Let K be a compact set in the plane, and let Ω be the unbounded component of its complement. If for any real valued function $f(\zeta_1, \zeta_2)$, defined and continuous on the boundary of Ω, there is a real valued function $\mu(x, y)$ harmonic in Ω, tending to a constant as (x, y) tends to infinity and tending to $f(\zeta_1^*, \zeta_2^*)$ as (x, y) tends to (ζ_1^*, ζ_2^*) from values in Ω, where (ζ_1^*, ζ_2^*) is an arbitrary point on the boundary of Ω, we say Ω is a Dirichlet domain and that K is a regular compact set.

LEMMA 3.3 [1, 2]. *Let* E *be a bounded Borel set in the plane of positive capacity* $C(E)$. *Then for any* ε, $0 < \varepsilon < C(E)$, *there is a regular compact set in* E *of capacity greater than* $C(E) - \varepsilon$.

LEMMA 3.4 [7]. *Let* μ *be an undetermined weight measure, let* K *be a regular compact subset of* $S(\mu)$, *let* E *be a carrier of* μ *and let* n *be a positive integer. Then*(a) *there is a non-negative Borel measurable function* $w_n(x)$ *with the property that* $\int w_n(x)\, d\mu = 1/n^2$ *and* $A_n = \{x : w_n(x) > 0\}$ *is a compact subset of* E, *and* (b) *there is a sequence of positive integers* $\{m_n\}$, $n = 1, 2, \ldots$, *with the property that* $\lim_{n \to \infty} m_n^{1/n} = 1$, *such that if* $Q_n(x)$ *is any polynomial of degree* n,

$$\int |Q_n(x)|^2 w_n(x)\, d\mu \geqslant \left(\frac{\| Q_n(x) \|_K}{2}\right)^2 \frac{1}{n^2 m_n},$$

where $\| Q_n(x) \|_K = \max_{x \in K} |Q_n(x)|$.

LEMMA 3.5 [5, p. 73]. *Let* K *be a compact set with capacity* $C(K)$, *and* *let* $T_n(z, K)$ *be the monic polynomial of degree* n *of least uniform norm on* K. *Then* $\lim_{n\to\infty}(\| T_n(z, K)\|_K)^{1/n} = C(K)$.

We now outline the proof, after which we carry out the details. In the hypothesis of Theorem 2.1 we are given a weight measure μ and constants C_1, C_2 such that $\underline{C} < C_1 < C_2 < \bar{C}$. By Lemma 3.2 we let E be a carrier of μ with $C(E) < C_1$, and with the representation $\bigcup_1^\infty E_n$ having the properties specified in the lemma. Let K be a regular compact subset of $S(\mu)$ with $C(K) > C_2$. Such a set exists by Lemma 3.3. We then chose two increasing sequences of integers $\{s_n\}$, $\{t_n\}$, $n = 1, 2,\ldots$, such that

$$\| T_{s_1}(x, E_1)\|_{E_1} \leqslant (C_1)^{s_1}, \tag{3.1}$$

$$\sum_{p=t_n}^{\infty} \frac{1}{p^2} \leqslant \left(\frac{C_1}{2}\right)^{2s_n}, \qquad n = 1, 2,\ldots, \tag{3.2}$$

and

$$\| T_{s_n}(x, H_n)\|_{H_n} \leqslant (C_1)^{s_n}, \qquad n = 2, 3,\ldots, \tag{3.3}$$

where $H_n = E_n \cup A_{t_1} \cup \cdots \cup A_{t_{n-1}}$, $n = 2, 3,\ldots.$ With these sequences we construct the function

$$w(x) = w^*(x) + w^{**}(x), \tag{3.4}$$

where

$$w^*(x) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\left(\frac{C_1}{2}\right)^{2s_{n-1}} - \left(\frac{C_1}{2}\right)^{2s_n}\right)\frac{\chi_{E_n}(x)}{\mu(E_n)}, \tag{3.5}$$

where $s_0 = 0$, and

$$w^{**}(x) = \frac{1}{2}\frac{\sum_{n=1}^\infty w_{t_n}(x)}{\sum_{n=1}^\infty 1/t_n^2}, \tag{3.6}$$

where $w_n(x)$ are the functions of Lemma 3.4 for the sets K and E already chosen in this discussion. We then show that $w(x) > 0$ a.e. μ, and $\int w(x)\, d\mu = 1$, so that by Lemma 3.1 $\nu = \mu_w$ is carrier to μ. The final step is to show that $\overline{\lim}\, \lambda_{s_n}(\nu) \leqslant C_1$ and $\underline{\lim}\, \lambda_{t_n}(\nu) \geqslant C_2$, to complete the proof of Theorem 2.1.

We begin supplying the details of this sketch by giving an informal proof that sequences exist satisfying (3.1), (3.2), and (3.3). Since this is a critical step of the proof, we supply a proof based in the principle of recursive definition [4, p. 10] at the end of this section. Since $C(E_1) \leqslant C(E) < C_1$, we can choose s_1 as the least integer for which (3.1) is satisfied. We then choose t_1 as the smallest integer satisfying (3.2) for $n=1$. Since $H_2 \subset E$, we can use Lemma 3.5 and choose s_2 as the smallest integer which satisfies (3.3) for $n=2$, and also satisfies $s_2 > s_1$. We then choose t_2 as the least integer which satisfies (3.2) for $n=2$ and satisfies $t_2 > t_1$. We proceed in this manner to construct the remainder of the sequences.

Since $w^*(x)$ is positive on E, the same is true of $w(x)$, and since E is a carrier of μ, we have $w(x) > 0$ a.e. μ. An easy calculation shows that $\int w(x)\,d\mu = 1$, so that $\nu = \mu_w$ is a weight measure and is carrier related to μ. We now consider estimates on $\lambda_n(\nu)$ for $n = t_k$, $k = 1, 2,...,$ Using Lemma 3.4 we find

$$\lambda_{t_k}^{2t_k}(\nu) = \int |P_{t_k}(x, \nu)|^2\,d\nu = \int |P_{t_k}(x)|^2\,w(x)\,d\mu$$

$$\geqslant \int |P_{t_k}(x)|^2\,w^{**}(x)\,d\mu \geqslant \frac{1}{2}\frac{1}{\sum_{n=1}^{\infty} 1/t_n^2} \int |P_{t_k}(x)|^2\,w_{t_k}(x)\,d\mu$$

$$\geqslant \frac{1}{2}\frac{1}{\sum_{n=1}^{\infty} 1/t_n^2}\frac{1}{t_k^2\,m_{t_k}}\left(\frac{\|P_{t_k}(x)\|_K}{2}\right)^2.$$

Since $\|P_{t_k}(x)\|_K \geqslant (C(K))^{t_k}$ [5, p. 62], we have

$$\underline{\lim}_{k\to\infty}\ \lambda_{t_k}(\nu) \geqslant C_2, \text{ so that } \overline{\lim}_{n\to\infty}\ \lambda_n(\nu) \geqslant C_2.$$

We now want to show that $\overline{\lim}_{k\to\infty} \lambda_{s_k}(\nu) \leqslant C_1$, which yields $\underline{\lim}\lambda_n(\nu) \leqslant C_1$. For k fixed and $\geqslant 2$, we have

$$\lambda_{s_k}^{2s_k} = \int |P_{s_k}(x, \nu)|^2\,d\nu \leqslant \int |T_{s_k}(x, H_k)|^2\,d\nu \tag{3.7}$$

$$\leqslant \int_{H_k} |T_{s_k}(x, H_k)|^2\,d\nu + \int_{E\setminus H_k} |T_{s_k}(x, H_k)|^2\,d\nu. \tag{3.8}$$

We use the notation that $\tilde{A}$ is the complement of A and $A\setminus B = A \cap \tilde{B}$. The second inequality in (3.7) uses the fact that if $Q_n(x)$ is a monic polynomial of degree n, then $\int |Q_n(x)|\,d\nu$ is least when $Q_n(x) = P_n(x, \nu)$. By (3.3), the first integral in (3.8) is bounded by $(C_1)^{2s_k}$. The second integral in (3.8) is equal to

$$\int_{E\setminus H_k} |T_{s_k}(x, H_k)|^2\,w^*(x)\,d\mu + \int_{E\setminus H_k} |T_{s_k}(x, H_k)|^2\,w^{**}(x)\,d\mu. \tag{3.9}$$

To bound the first integral in (3.9) we observe that $E\setminus H_k \subset \tilde{E}_k$ and for

$x \in I$, $|T_{s_k}(x, H_k)| \leqslant 2^{s_k}$ since the zeros of $T_{s_k}(x, H_k)$ lie in I. Thus we have the bound

$$2^{2s_k} \int_{\tilde{E}_k} w^*(x)\, d\mu = \frac{2^{2s_k}}{2} \sum_{n=1}^{\infty} \int_{\tilde{E}_k} \left(\left(\frac{C_1}{2}\right)^{2s_{n-1}} - \left(\frac{C_1}{2}\right)^{2s_n} \right) \frac{\chi_{E_n}(x)}{\mu(E_n)}\, d\mu$$
$$\leqslant \frac{2^{2s_k}}{2} \sum_{n=k+1}^{\infty} \left(\left(\frac{C_1}{2}\right)^{2s_{n-1}} - \left(\frac{C_1}{2}\right)^{2s_n} \right) = \frac{2^{2s_k}}{2} \left(\frac{C_1}{2}\right)^{2s_k} = \frac{1}{2} (C_1)^{2s_k}.$$

We bound the second integral in (3.9) in similar fashion. Since $E \backslash H_k \subset \widetilde{\bigcup_{n=1}^{k-1} A_{t_n}}$ and $|T_{s_k}(x, H_k)| \leqslant 2^{s_k}$ for $x \in I$, if we let $M = 1/\sum_{n=1}^{\infty} 1/t_n^2$, then we have the bound

$$\frac{(2^{2s_k})M}{2} \int \sum_{n=1}^{\infty} w_{t_n}(x)\, d\mu \leqslant \frac{(2^{2s_k})M}{2} \sum_{n=k}^{\infty} \int w_{t_n}(x)\, d\mu\, \widetilde{\bigcup_{n=1}^{k-1}} A_{t_n}$$
$$= \frac{(2^{2s_k})M}{2} \sum_{n=k}^{\infty} \frac{1}{t_n^2} \leqslant \frac{(2^{2s_k})M}{2} \sum_{n=t_k}^{\infty} \frac{1}{n^2} \leqslant \frac{M}{2} (C_1)^{2s_k},$$

where (3.2) is used in the last inequality. Thus

$$\lambda_{s_k}^{2s_k} \leqslant \left(\frac{1+M}{2}\right) C_1^{2s_k}$$

and $\overline{\lim}\, \lambda_{s_k} \leqslant C_1$, finishing the proof of Theorem 2.1.

The principle of recursive definition states that if f_n is a mapping from R^n to R, $R = (-\infty, \infty)$, and x_1 is given in R, then there exists a unique sequence $\{x_n\}$, $n = 1, 2, \ldots$ such that $x_{n+1} = f_n(x_1, \ldots, x_n)$. For a compact set $K \subset I$, and $C_1 > C(K)$, let $\{K, C_1\}$ be the least positive integer for which $\| T_n(x, K) \|_K^{1/n} \leqslant C_1$ for all $n \geqslant \{K, C_1\}$. This integer exists by Lemma 3.5. Now let

$$f_{2n-1} = \left(\left[\left(\frac{2}{C_1}\right)^{2x_{2n-1}} \right] + 2 \vee x_{2n-2} + 1 \right), \qquad n = 1, 2, \ldots, \quad (3.10)$$

and

$$f_{2n-2} = \left(\left\{ E_n \cup \bigcup_{k=1}^{n-1} A_{x_{2k}}, C_1 \right\} \vee x_{2n-3} + 1 \right), \qquad n = 2, 3, \ldots, \quad (3.11)$$

where $[\alpha]$ is the greatest integer in α and $(a \vee b)$ is the maximum of the real numbers a and b. Then if x_1 is given by $\{E_1, C_1\}$, there is a unique sequence $\{x_k\}$, $k = 1, 2, \ldots$, which satisfies $x_{2n} = f_{2n-1}(x_1, \ldots, x_{2n-1})$, $n = 1, 2, \ldots$, and $x_{2n-1} = f_{2n-2}(x_1, \ldots, x_{2n-2})$, $n = 2, 3, \ldots$.

Let $x_{2n-1} = s_n$, $n = 1, 2, \ldots$, let $x_{2n} = t_n$, $n = 1, 2, \ldots$, and let $H_n = E_n \cup \bigcup_{k=1}^{n-1} A_{t_k}$, $n = 2, 3, \ldots$. We see in this change of notation that

$$s_1 = \{E_1, C_1\}, \tag{3.12}$$

$$t_n = \left(\left[\left(\frac{2}{C_1}\right)^{2s_n}\right] + 2 \vee t_{n-1} + 1\right), \qquad n = 1, 2, \ldots, \tag{3.13}$$

$$s_n = (\{H_n, C_1\} \vee s_{n-1} + 1), \qquad n = 2, 3, \ldots. \tag{3.14}$$

From (3.12) we see that $\| T_{s_1}(x, E_1) \|^{1/s_1} \leqslant C_1$. From (3.14) we see that $\{s_n\}$ $n = 1, 2, \ldots$, is an increasing sequence of integers and that $\| T_{s_n}(x, H_n) \|_{H_n}^{1/s_n} \leqslant C_1$.

From (3.13) we see that $\{t_n\}$, $n = 1, 2, \ldots$, is an increasing sequence of integers. Further we note that $t_n \geqslant [(2/C_1)^{2s_n}] + 2 \geqslant (2/C_1)^{2s_n} + 1$. Hence $t_n - 1 \geqslant (2/C_1)^{2s_n}$ and $\frac{1}{t_n - 1} \leqslant (C_1/2)^{2s_n}$. Thus $\sum_{p=t_n}^{\infty} \frac{1}{p^2} < \int_{t_n-1}^{\infty} dt/(t_n - 1) < (C_1/2)^{2s_n}$, $n = 1, 2, \ldots$, and the sequences $\{s_n\}$, $\{t_n\}$ satisfy all the conditions of (3.1), (3.2), and (3.3).

4. Proof of Theorem 2.2

The first thing we show is that $N_n = N_n(\mu)$ is nonincreasing. In fact

$$(N_n(\mu))^2 = \int |P_n(x, \mu)|^2 \, d\mu \leqslant \int |xP_{n-1}(x, \mu)|^2 \, d\mu$$
$$\leqslant \int |P_{n-1}(x, \mu)|^2 \, d\mu = N_{n-1}^2(\mu).$$

We want to show that if α and β are limit points of $\lambda_n = \lambda_n(\mu)$, and $\alpha < \gamma < \beta$, then γ is also a limit point of λ_n. We first consider the case in which $\underline{\lim}_{n \to \infty} \lambda_n = \delta > 0$, and then reduce the general case to this case.

Now $\log \lambda_n - \log \lambda_{n+1} = (1/n) \log N_n - (1/(n+1)) \log N_{n+1}$, so that $(n+1)(\log \lambda_n - \log \lambda_{n+1}) = (1 + (1/n)) \log N_n - \log N_{n+1} = \log N_n / N_{n+1} + \log N_n^{1/n} \geqslant \log \delta/2$ for sufficiently large n, say $n \geqslant n^*$. Thus $\log \lambda_{n+1} \leqslant \log \lambda_n + (\log(2/\delta)/(n+1))$.

Since $\int |P_n(x, \mu)|^2 \, d\mu \leqslant \int |T_n(x, I)|^2 \, d\mu \leqslant (1/2^{n-1})^2$ from known results about Tchebycheff polynomials, it follows that $\overline{\lim}_{n \to \infty} \lambda_n \leqslant \frac{1}{2}$, so that $\delta \leqslant \frac{1}{2}$ and $\log 2/\delta > 0$.

Let n_1 be the first integer greater or equal to n^* for which $\log \lambda_{n_1} \leqslant \log \gamma$, and let n_2 be the first integer greater than n_1 for which $\log \lambda_{n_2} > \log \gamma$. Then $\log \lambda_{n_2 - 1} \leqslant \log \gamma < \log \lambda_{n_2}$ and $\log \gamma - \log \lambda_{n_2 - 1} \leqslant \log \lambda_{n_2} - \log \lambda_{n_2 - 1} \leqslant \log (2/\delta) | n_2$. Let n_3 be the first integer after n_2 for which $\lambda_{n_3} \leqslant \log \gamma$, and let n_4 be the next integer after n_3 for which $\log \lambda_n > \log \gamma$. Then as before $\log \gamma - \log \lambda_{n_4 - 1} \leqslant \log 2/\delta/n_4$. The process yields an infinite sequence since $\alpha < \gamma < \beta$ and $\lim_{k \to \infty} \lambda_{n_{2k} - 1} = \gamma$.

If $\underline{\lim}_{n\to\infty}\lambda_n=0$ and α and β are limit points of λ_n we can reduce the analysis to the previous case as follows. Say γ is such that $\alpha<\gamma<\beta$. Choose ε so that $0<\varepsilon<\gamma$ and consider the new sequence $N_n^*=N_n+\varepsilon^n$, and the derived sequence $\lambda_n^*=(N_n^*)^{1/n}$. Since N_n^* is nonincreasing we proceed as before, and note that $\underline{\lim}_{n\to\infty}\lambda_n^*=\varepsilon$, $\overline{\lim}_{n\to\infty}\lambda_n^*\geqslant\beta$. Hence there is a subsequence for which $\lim_{n\to\infty}\lambda_{k_n}^*=\gamma$ and it is finally verified that $\lim_{n\to\infty}\lambda_{k_n}=\gamma$ to complete the demonstration.

5. Discussion

Theorem 2.2 has a consequence for zero distribution because of Theorem 5.1 which we state here after necessary preliminaries, and which we will prove in a forthcoming paper.

Definition 5.1. Let μ be a weight measure, and let n be a positive integer. The zeros of $P_n(x,\mu)$ are simple and lie in I. Let ν_n be a unit measure having mass $1/n$ at each zero of $P_n(x,\mu)$. We call ν_n a zero measure of μ.

Definition 5.2. Let μ be a weight measure and let $\{\nu_{k_n}\}$, $n=1,2,\ldots$, be a sequence of zero measures of μ. If there is a Borel measure ν such that $\lim_{n\to\infty}\int f(x)\,d\nu_{k_n}=\int f(x)\,d\nu$ for all functions $f(x)$ continuous on I, we say $\{\nu_{k_n}\}$ converges, and that it converges to ν. This is also called weak*convergence. We call ν a zero distribution measure of μ.

Lemma 5.1 [3, p. 290]. *If μ is a weight measure and $\{\nu_{k_n}\}$, $n=1,2,\ldots$, is a sequence of zero measures of μ, then some further subsequences of these zero measures converges.*

Theorem 5.1. *Let μ be a weight measure with $\underline{C}>0$. Then if a sequence of zero measures of μ converges, say $\{\nu_{k_n}\}$ converges to ν, then the corresponding sequence of linearized norms $\{\lambda_{k_n}\}$, converges, say to λ, and if any other sequences of zero measures of μ converges to ν, the corresponding sequence of linearized norms converges to the same number λ.*

Let μ be a norm oscillatory weight measure with $\underline{C}>0$. Because of Theorem 2.2, the sequence $\{\lambda_n(\mu)\}$, $n=1,2,\ldots$, has uncountable many limit values. Let λ_1,λ_2 be two distinct values with $\lim_{n\to\infty}\lambda_{s_n}(\mu)=\lambda_1$ and $\lim_{n\to\infty}\lambda_{t_n}(\mu)=\lambda_2$. By Lemma 5.1, there is a subsequence, say $\{p_n\}$ of the sequence $\{s_n\}$ for which ν_{p_n} converges, say to ν_1, and a subsequence, say $\{q_n\}$, of the sequence $\{t_n\}$ for which ν_{q_n} converges, say to ν_2. Since $\lim_{n\to\infty}\lambda_{p_n}=\lambda_1$ and $\lim_{n\to\infty}\lambda_{q_n}=\lambda_2$, it follows from Theorem 5.1 that

$v_1 \neq v_2$. Thus a norm oscillatory weight measure with $\underline{C} > 0$ has incountably many zero distribution measures associated with it.

We finally announce a theorem that characterizes the possible limit values of the linearized norms of undetermined measures.

THEOREM 5.2. *Let μ be an undetermined measure with $\underline{C}$ and $\bar{C}$ as lower and upper carrier capacities. Let α, β be any numbers satisfying $\underline{C} \leqslant \alpha \leqslant \beta \leqslant \bar{C}$. Then there is a weight measure v, carrier related to μ, such that the interval of limit points of the sequence $\{\lambda_n(v)\}$, $n = 1, 2, \ldots$, is precisely $[\alpha, \beta]$.*

It will be shown in a future publication that the inequality $\underline{C} < C_1$ in Theorem 2.1 can only be improved to $\underline{C} \leqslant C_1$, and that the inequality $\underline{C} \leqslant \alpha$ in Theorem 5.2 cannot be improved. In other words, we will show that for any weight measure v carrier related to a weight measure μ, $\underline{\lim}_{n \to \infty} \lambda_n(v) \geqslant \underline{C}$ where $\underline{C}$ is the lower carrier capacity of μ.

REFERENCES

1. A. ANCONA, Démonstration d'une conjecture sur la capacité et l'effilement, *C. R. Acad. Sci. Paris Ser. I* **297** (1983), 393–395.
2. A. ANCONA, Sur une conjecture concernant la capacité et l'effilement, *in* "Colloque du Théorie du Potential," Jacques Deny, 1984, to appear in Springer-Verlag Lecture Notes.
3. P. BILLINGSLEY, "Probalility and Measure," Wiley, New York, 1979.
4. H. L. ROYDEN, "Real Analysis," 2nd ed., Macmillan, New York, 1968.
5. M. TSUJI, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
6. J. L. ULLMAN, On the regular behavior of orthogonal polynomials, *Proc. London Math. Soc.* (3) **24** (1972), 119–148.
7. J. L. ULLMAN, Orthogonal polynomials for general measures. II *in* "Conference in Honor of Edmond Laquerre," Bar-le-Duc, 1984, to appear in Springer-Verlag Lecture Notes.

Printed in Belgium